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APPLICATION OF VARIATIONAL METHODS TO TRANSONIC FLOWS

WITH SHOCK WAVES

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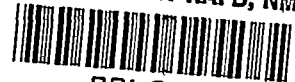APPLICATION OF VARIATIONAL METHODS TO TRANSONIC FLOWS

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SUMMARY

Variational methods for the approximate solution of subsonic and transonic flows of a compressible fluid before the occurrence of shock waves have been carried out in previous papers. The methods fail as soon as the shock waves occur as the flow behind the shock waves now becomes rotational and has variable entropy. Since most transonic flows are accompanied by shock waves, a method which allows for shock waves and variable entropy is necessary for the study of such flows. By modifying Bateman's variational principle for irrotational flows, it is shown that a variational principle for flows with rotation and variable entropy can be obtained. By applying this variational principle to the regions of flow behind shock waves and Bateman's original principle to the other regions in the fluid, shock equations can be directly obtained. A procedure for computing numerical solutions for such flows is suggested, and a numerical example is carried out. At high Mach number above a certain limiting value, the results show that irrotational flow fails. However, by inserting shock waves and allowing a part of the flow to be rotational, computation indicates that solution exists again.

INTRODUCTION

In previous papers (references 1 to 7) the senior author and his associates have succeeded in applying the variational methods to the study of subsonic and transonic flows of a compressible fluid past arbitrary bodies before the occurrence of shock waves. Numerical examples were carried out in the case of the flow past a circular cylinder, an elliptical cylinder, a Kaplan bump, a sphere, and an ellipsoid. The results were found to check excellently with those computed by other methods. These results indicate that the variational method will give good approximations to flows past either thick or thin bodies and at both low and high Mach numbers. The method as formulated, however, can only be applied to irrotational flows. As soon as shock waves occur the method fails because the flow behind the shocks then becomes rotational. As most transonic flows are accompanied by shock waves, different methods

which allow the rotationality of the flow as well as variable entropy must be formulated if any significant results are to be obtained from the studies of such flows.

A reexamination of Bateman's variational prin.iples (reference 8) indicates that with some modification the principle in terms of the stream function ψ, which was originally formulated for irrotational flows, can also be applied to rotational flows with entropy change. The resulting variational integral is the same as the one recently obtained by Lin and Rubinov (reference 9). In the study of transonic flows this integral is to be applied in the region of flow after shock waves and the original Bateman integral in the other regions. It can be shown that the shock equations are directly obtainable from these principles. With the variational principles obtained, a direct method for the approximate solution of transonic flow with shock waves may again be formulated following the Rayleigh-Ritz procedure. The actual carrying out of such a method however was found too laborious. Instead, Galerkin's method may be used which shortens the numerical work to a great extent. In this report approximate solution in the case of the transonic flow past a circular cylinder has been carried out. The results show that when the Mach number increases to a certain limiting value without allowing for shock waves the variational method does not have a solution. This probably indicates the breakdown of the irrotational flow. By allowing the occurrence of shock waves, solution again exists.

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SYMBOLS

a	velocity of sound
$A_{mn}, B_{mn}, A'_{mn}, B'_{mn}$	undetermined parameters
c_p, c_v	specific heat at constant pressure and volume, respectively
c_1, c_2, c_3	constants
M	Mach number (q/a)
n	normal distance

p	pressure
q	magnitude of velocity vector, with components u and v
S	entropy
u	velocity component in x-direction
v	velocity component in y-direction
x,y	coordinates of a point in the fluid
γ	isentropic exponent $\left(c_p/c_v\right)$
φ	velocity potential function
ψ	stream function
ρ	mass density
ω	rotation of flow
U	velocity of undisturbed stream

Subscripts

x,y,r,θ	denote differentiation in corresponding direction
s	isentropic stagnation conditions for undisturbed stream
o	conditions of undisturbed stream

GOVERNING DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

Consider a steady two-dimensional flow passing a cylinder of arbitrary shape. Most aerodynamic problems are concerned with either flow from rest or flow which is parallel and uniform at infinity. In such cases before the occurrence of shock waves the flow is irrotational. As the speed of the flow increases, shock waves occur. For example, the flow past an airfoil is shown in figure 1. The flow is assumed to be subsonic at infinity. Because of the presence of the airfoil, if the Mach number of the flow is sufficiently high there will be a region of supersonic flow near the surface of the airfoil, as shown by the dotted

lines in the figure. Shock waves will occur at some points on the
surface in these supersonic regions and will terminate in the flow
where local Mach number is equal to 1. Then in the region extending
to infinity behind the shock waves and bounded by the stream lines
passing through the points where the shock terminates, the flow is
rotational and with variable entropy. For convenience, this region
will be denoted by D_2, and the region outside D_2, by D_1. Then
in D_1 the following equations must be satisfied:

(1) The two equations of motion

$$uu_x + vu_y = -p_x/\rho$$

$$uv_x + vv_y = -p_y/\rho$$

(1)

where u and v are the velocity components in the x- and y-directions,
ρ the density, and p the pressure, and the subscripts x and y
indicate partial differentiation in the corresponding direction.

(2) The equation of continuity

$$(\rho u)_x + (\rho v)_y = 0$$

(2)

(3) The equation of state

$$p = c_1 \rho^\gamma$$

(3)

where c_1 is a constant and γ is the ratio of specific heats. If the
subscript s is used to indicate the stagnation conditions, $c_1 = p_s/\rho_s^\gamma$.

(4) The irrotationality condition

$$u_y - v_x = 0$$

(4)

With aid of equation (4), equation (1) can be integrated to give

$$\frac{p}{(\gamma - 1)\rho} + \frac{q^2}{2} = c_2 \tag{5}$$

where $q^2 = u^2 + v^2$ and c_2 is another constant.

The boundary conditions are that at infinity the velocity components u and v are equal to the given values, and on the solid boundary of the airfoil the normal component of the velocity vanishes.

In region D_2 the flow becomes now rotational and the entropy S is no longer a constant. Here the governing differential equations are as follows: Equation of motion (1) and the equation of continuity (2) are still valid. The equation of state has to be changed to the following form:

$$p = c_3 e^{S/c_v} \rho^\gamma \tag{6}$$

where c_v is the specific heat at constant volume and where c_3 is a constant. The energy equation is

$$u S_x + v S_y = 0 \tag{7}$$

For flows derived from isentropic irrotational flows by a shock, there is the so-called condition of isoenergetic flow

$$\frac{\gamma p}{(\gamma - 1)\rho} + \frac{q^2}{2} = c_2 \tag{5a}$$

where c_2 is a constant throughout the flow, before and after the shocks. This equation is then identical to equation (5); it implies equation (7) when equations (1) and (2) are satisfied.

The boundary condition at the solid body is the same as before; that is, the normal component of velocity must be zero. The condition at infinity, however, is no longer the same, because if the velocity were

constant at infinity the flow would again be irrotatiqnal. The correct boundary condition in this case is that the pressure must be a constant at infinity.

Since the flow in a part of the domain is$^{\circ}$ rotational, the velocity potential φ obviously cannot be used and it is convenient to introduce in such cases the stream function ψ defined by

$$\left.\begin{array}{c} \rho u = \psi_y \\[2mm] \rho v = -\psi_x \end{array}\right\} \tag{8}$$

Equation (7) indicates that the entropy S is a constant along each streamline. The entropy S can therefore be written in the following form:

$$S = c_v f(\psi) \tag{9}$$

and the equation of state (6) as

$$p = c_3 e^{f(\psi)} \rho^\gamma \tag{10}$$

With some calculation, the rotation of the flow may be written in terms of ψ as follows:

$$\begin{aligned} \omega &= v_x - u_y \\[2mm] &= c_3 \rho^\gamma e^{f(\psi)} f'(\psi) \big/ (\gamma - 1) \end{aligned} \tag{11}$$

VARIATIONAL PRINCIPLES

Instead of studying the boundary-value problems as formulated in the preceding paragraphs, it is sometimes more convenient to study the associated variational problems, especially when the exact solutions of the differential equations are very difficult to obtain. In such cases,

approximate methods of solution such as the Rayleigh-Ritz method and the Galerkin method will be extremely useful.

For irrotational flows, two variational principles were given by Bateman (reference 8), one in terms of the velocity potential and the other in terms of the stream function. The first one is more suitable for studying the flow passing arbitrary bodies and has been used by the senior author and his associates in computing such flows in references 1 to 7. In the present case, the second principle which is expressed in terms of ψ should be used.

If the pressure is

$$p = F(\rho) - \rho F'(\rho) \tag{12}$$

where $F(\rho)$ is some function of ρ and the prime indicates differentiation, the variational integral is

$$J_1 = \iint \left[F(\rho) + \frac{\left(\psi_x^2 + \psi_y^2 \right)}{2\rho} \right] dx\, dy \tag{13}$$

in which ρ and ψ are to be varied independently.

Equation (12) is a differential equation of the Clairaut type. If $p = c_1 \rho^\gamma$, the solution of equation (12) is $F(\rho) = c_2 \rho - \dfrac{c_1 \rho^\gamma}{\gamma - 1}$ The integral J_1 then becomes

$$J_1 = \iint \left[c_2 \rho - \frac{c_1 \rho^\gamma}{\gamma - 1} + \frac{\left(\psi_x^2 + \psi_y^2 \right)}{2\rho} \right] dx\, dy \tag{14}$$

The condition that $\delta J_1 = 0$ then leads to

$$\iint \left[\left(c_2 - \frac{c_1 \gamma}{\gamma - 1} \rho^{\gamma-1} \right) - \frac{1}{2\rho^2} \left(\psi_x^2 + \psi_y^2 \right) \right] \delta \rho \, dx\, dy -$$

$$\iint \left[\left(\psi_x/\rho \right)_x + \left(\psi_y/\rho \right)_y \right] \delta \psi \, dx\, dy + \int \frac{1}{\rho} \frac{\partial \psi}{\partial n} \delta \psi \, ds = 0$$

where the last integral is taken on the boundary of the domain. If this
line integral is zero, the condition $\delta J_1 = 0$ leads to the following
equations

$$\frac{1}{2\rho^2}\left(\psi_x{}^2 + \psi_y{}^2\right) + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = c_2$$

$$\left(\psi_x/\rho\right)_x + \left(\psi_y/\rho\right)_y = 0$$

as the Euler equations in the calculus of variation. The first equation
is the Bernoulli equation and the second is the condition of irrotation-
ality. The condition of continuity is satisfied automatically by intro-
ducing the stream function ψ. Thus if the line integral is equal to
zero the vanishing of the first variation of J leads to the desired
differential equations. If the line integral is not zero, as in similar
cases discussed in references 1, 2, 6, and 10, the value of the integral
must be subtracted from δJ_1 and J_1 should be modified accordingly.

It is interesting to note that although the variational integral (13)
is formulated by Bateman for irrotational flow only, it is also valid
for rotational flow if in equation (12) $F(\rho)$ is replaced by $F(\rho,S)$
and is solved from the correct equation of state, equation (6) or (10).
In this case

$$F - \rho\,\frac{\partial F}{\partial \rho} = c_3 e^{f(\psi)}\rho^\gamma$$

The solution of this differential equation may be written as

$$F(\rho,S) = c_2\rho - \frac{c_3 e^{f(\psi)}\rho^\gamma}{(\gamma - 1)} \tag{15}$$

and a variational integral for the rotational flow after shock waves may
be written as follows

$$J_2 = \iint \left\{\left[c_2\rho - \frac{c_3 e^{f(\psi)}\rho^\gamma}{(\gamma - 1)}\right] + \frac{\left(\psi_x{}^2 + \psi_y{}^2\right)}{2\rho}\right\} dx\ dy \tag{16}$$

This is the same integral recently obtained by Lin and Rubinov (reference 9). As before, the condition $\delta J_2 = 0$ leads to

$$\iint \left[c_2 - \frac{\gamma c_3 e^{f(\psi)} \rho^{\gamma-1}}{(\gamma - 1)} - \frac{\left(\psi_x^2 + \psi_y^2\right)}{2\rho^2} \right] \delta\rho \; dx \; dy -$$

$$\iint \left[\frac{c_3 \rho^\gamma e^{f(\psi)} f'(\psi)}{(\gamma - 1)} + \left(\psi_x / \rho\right)_x + \left(\psi_y / \rho\right)_y \right] \delta\psi \; dx \; dy +$$

$$\int \frac{1}{\rho} \frac{\partial \psi}{\partial n} \delta\psi \; ds = 0$$

The last integral is again taken on the boundary of the domain. If this integral is zero, there result

$$\frac{\left(\psi_x^2 + \psi_y^2\right)}{2\rho^2} + \frac{\gamma p}{(\gamma - 1)\rho} = c_2$$

$$\left(\psi_x / \rho\right)_x + \left(\psi_y / \rho\right)_y = -c_3 \frac{\rho^\gamma e^{f(\psi)}}{\gamma - 1} f'(\psi) \tag{17}$$

as the Euler's equations. The first equation is the energy equation (5a) and the second equation is the rotation equation (11). With the aid of the second equation, the equations of motion (1) can be derived from the first equation as follows: Writing the first equation in a slightly different form,

$$\frac{\gamma}{\gamma - 1} c_3 e^{f(\psi)} \rho^{\gamma-1} + \frac{1}{2}\left(u^2 + v^2\right) = c_2$$

Differentiating it with respect to x,

$$\gamma c_3 e^{f(\psi)} \rho^{\gamma-2} \rho_x + \frac{\gamma}{\gamma - 1} c_3 e^{f(\psi)} \rho^{\gamma-1} f'(\psi) \psi_x +$$

$$u u_x + v v_x = 0 \tag{18}$$

Since $p = c_3 e^{f(\psi)} \rho^\gamma$,

$$p_x/\rho = \gamma c_3 e^{f(\psi)} \rho^{\gamma-2} \rho_x + c_3 e^{f(\psi)} \rho^{\gamma-1} f'(\psi) \psi_x$$

Substitution in equation (18) results in

$$\frac{p_x}{\rho} + \frac{1}{\gamma-1} c_3 \rho^\gamma e^{f(\psi)} f'(\psi) \frac{1}{\rho} \psi_x + u u_x + v v_x = 0 \qquad (19)$$

As $c_3 \rho^\gamma e^{f(\psi)} f'(\psi) \big/ (\gamma-1) = v_x - u_y$ and $\frac{1}{\rho} \psi_x = v$, equation (19) is identical to the equation of motion in the x-direction. Similarly, differentiation of equation (17) with respect to y results in the other equation of motion.

SHOCK CONDITIONS AS A RESULT OF VARIATIONAL PRINCIPLE

In discussing the mathematical formulation of the problem, the flow as shown in figure 1 may be put into a simplified form as shown in figure 2, where D_1 and D_2 denote the regions when the flow is irrotational and rotational, respectively, and C_{1-2} is the shock wave plus the common boundary. The variational principle is that

$$\delta J = \delta J_1 + \delta J_2 = 0$$

leads to the desired differential equations in the corresponding domain. Here

$$J_1 = \iint_{D_1} \left[c_2 \rho_1 - \frac{c_1 \rho_1{}^\gamma}{\gamma-1} + \frac{\left(\psi_{1x}{}^2 + \psi_{1y}{}^2\right)}{2\rho_1} \right] dx\, dy$$

$$J_2 = \iint_{D_2} \left[c_2 \rho_2 - \frac{c_3 e^{f\left(\psi_2\right)} \rho_2{}^\gamma}{(\gamma-1)} + \frac{\left(\psi_{2x}{}^2 + \psi_{2y}{}^2\right)}{2\rho_2} \right] dx\, dy \qquad (20)$$

where as shown in the above expressions J_1 is integrated over D_1 and J_2 over D_2; ρ_1, ψ_1 and ρ_2, ψ_2 indicate that ρ and ψ are different functions in the domains D_1 and D_2; and the functions ρ_1, ρ_2, ψ_1, and ψ_2 are to be varied independently. The condition $\delta J = 0$ therefore leads to

$$\iint_{D_1}\left[c_2 - \frac{\gamma c_1 \rho_1^{\gamma-1}}{(\gamma-1)} - \frac{\left(\psi_{1x}^2 + \psi_{1y}^2\right)}{2\rho_1^2}\right]\delta\rho_1 \ dx \ dy \ -$$

$$\iint_{D_1}\left[\left(\psi_{1x}/\rho_1\right)_x + \left(\psi_{1y}/\rho_1\right)_y\right]\delta\psi_1 \ dx \ dy \ +$$

$$\iint_{D_2}\left[c_2 - \frac{\gamma c_3 e^{f(\psi_2)}\rho_2^{\gamma-1}}{(\gamma-1)} - \frac{\left(\psi_{2x}^2 + \psi_{2y}^2\right)}{2\rho_2^2}\right]\delta\rho_2 \ dx \ dy \ -$$

$$\iint_{D_2}\left[\frac{c_3\rho_2^{\gamma}e^{f(\psi_2)}f'(\psi_2)}{(\gamma-1)} + \left(\psi_{2x}/\rho_2\right)_x + \left(\psi_{2y}/\rho_2\right)_y\right]\delta\psi_2 \ dx \ dy \ +$$

$$\int_{C_1}\frac{1}{\rho_1}\frac{\partial\psi_1}{\partial n}\delta\psi_1 \ ds \ +$$

$$\int_{C_2}\frac{1}{\rho_2}\frac{\partial\psi_2}{\partial n}\delta\psi_2 \ ds \ + \int_{C_{1-2}}\left(\frac{1}{\rho_1}\frac{\partial\psi_1}{\partial n}\delta\psi_1 - \frac{1}{\rho_2}\frac{\partial\psi_2}{\partial n}\delta\psi_2\right) \ ds \ = \ 0$$

Consider the line integrals first. On a part of C_1 and C_2 where the boundary is the surface of the body, ψ_1 and ψ_2 are equal to the chosen constants. Then $\delta\psi_1 = \delta\psi_2 = 0$, and the first two line integrals are zero. These integrals may not be zero when taken over the other parts of the boundaries. In such cases, the values of the integrals must be subtracted from δJ and J must be modified accordingly. On the shock waves, if

$$\left(1/\rho_1\right)\left(\partial\psi_1/\partial n\right) = \left(1/\rho_2\right)\left(\partial\psi_2/\partial n\right) \tag{21}$$

and

$$\psi_1 = \psi_2 \tag{22}$$

the third integral vanishes. Condition (21) requires the velocity components tangent to a shock wave to be the same in passing through the shock. Condition (22) is the continuity equation in crossing a shock wave. On the common boundary, since it is a streamline, $\delta\psi_1 = \delta\psi_2 = 0$ and the line integral vanishes.

When the line integrals are zero or modified to be zero, the following Euler equations are obtained from the well-known rules of the Calculus of Variation. They are

$$\frac{\gamma p_1}{(\gamma - 1)\rho_1} + \frac{\left(\psi_{1x}^{~2} + \psi_{1y}^{~2}\right)}{2\rho_1^{~2}} = c_2 \quad \text{in} \quad D_1 \tag{23}$$

$$\frac{\gamma p_2}{(\gamma - 1)\rho_2} + \frac{\left(\psi_{2x}^{~2} + \psi_{2y}^{~2}\right)}{2\rho_2^{~2}} = c_2 \quad \text{in} \quad D_2 \tag{24}$$

$$\frac{\partial}{\partial x}\!\left(\frac{1}{\rho_1}\,\psi_{1x}\right) + \frac{\partial}{\partial y}\!\left(\frac{1}{\rho_1}\,\psi_{1y}\right) = 0 \quad \text{in} \quad D_1 \tag{25}$$

$$\frac{\partial}{\partial x}\!\left(\frac{1}{\rho_2}\,\psi_{2x}\right) + \frac{\partial}{\partial y}\!\left(\frac{1}{\rho_2}\,\psi_{2y}\right) = -c_3 \rho_2^{~\gamma} e^{f(\psi_2)} f'(\psi_2)\Big/(\gamma - 1) \quad \text{in} \quad D_2 \tag{26}$$

Equations (23) and (24) indicate that the energy in the fluid remains unchanged in crossing a shock wave. With equations (23) and (25) and equations (24) and (26), the equation of motion can be derived; that is, the momentum of the flow also remains the same in crossing a shock. The continuity equation is already given as equation (22). Therefore, from

SUBSONIC AND TRANSONIC FLOWS PAST A CIRCULAR CYLINDER

BEFORE OCCURRENCE OF SHOCK WAVES

As a first step to the solution of transonic flows with shock waves, the flow before the occurrence of shock waves will be considered. The variational integral J contains two variables, ρ and ψ. Numerical calculation may be carried out by following the Rayleigh-Ritz procedure. Unlike the cases considered in references 1 to 6 where the velocity potential φ is used, the labor involved in the computation by taking ψ and ρ as variables in the Rayleigh-Ritz method becomes excessively large. Instead, Galerkin's method was found to be much more simple in this case. The application of Galerkin's method to compressible-flow problems has been discussed in detail in references 3 and 4 and can be briefly outlined as follows: First the variables ψ and ρ are written in the form of series which satisfy the boundary conditions but with undetermined parameters, such as

$$\left.\begin{aligned} \psi &= \psi_0 + \sum_m \sum_n A_{mn}\,\psi_{mn} \\[2em] \rho &= \rho_0 + \sum_m \sum_n B_{mn}\rho_{mn} \end{aligned}\right\} \qquad (27)$$

where A_{mn} and B_{mn} are the undetermined parameters. Then A_{mn} and B_{mn} are determined from the conditions

$$\iint \left[\left(\psi_x/\rho\right)_x + \left(\psi_y/\rho\right)_y\right]\frac{\partial \psi}{\partial A_{mn}}\,dx\,dy = 0 \qquad (28)$$

$$\iint \left[\left(c_2\rho^2 - \frac{c_1\gamma}{\gamma - 1}\,\rho^{\gamma+1}\right) - \frac{1}{2}\left(\psi_x^2 + \psi_y^2\right)\right]\frac{\partial \rho}{\partial B_{mn}}\,dx\,dy = 0 \qquad (29)$$

With A_{mn} and B_{mn} determined, equations (27) give an approximate solution to the problem.

It was found, however, that near the limiting Mach number where shock waves are about to occur, the ρ-series converges rather slowly and

many parameters have to be used. The amount of work involved in the integration and in the solution of the resulting simultaneous equations with many parameters again becomes very great. A modified Galerkin method has been tried and it was found possible to reduce much of the numerical work. The method is essentially as follows: Instead of determining B_{mn} by Galerkin's method, these parameters can be solved in terms of A_{mn} from the Bernoulli equation:

$$\left(c_2\rho^2 - \frac{c_1\gamma}{\gamma - 1}\rho^{\gamma+1}\right) - \frac{1}{2}\left(\psi_x{}^2 + \psi_y{}^2\right) = 0 \tag{30}$$

by the method of equal coefficients. The parameters A_{mn} then become the only unknowns and they are determined by the Galerkin method.

In the case of two-dimensional flow past a circular cylinder with unit radius the boundary conditions are at $r = \infty$:

$$\rho = \rho_0$$

$$\left(\psi_\theta/\rho_0 r\right)^2 + \left(\psi/\rho_0\right)^2 = U^2$$

and at $r = 1$

$$\psi_\theta = 0$$

Consider the case where the circulation is zero. The flow must then be symmetrical with respect to both the x- and y-axis. The boundary conditions and the conditions of symmetry are satisfied if ψ and ρ are assumed in series of the following forms:

$$\psi = \rho_0 U\left[\left(r - \frac{1}{r}\right)\sin\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left(\frac{1}{r^m} - \frac{1}{r^{m+2}}\right)\sin n\theta\right] \tag{31}$$

and

$$\rho = \rho_0\left(1 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}\frac{\cos 2n\theta}{r^{2m}}\right) \tag{32}$$

Taking two parameters in ψ, equation (31) becomes

$$\psi = \rho_o U\left[\left(r - \frac{1}{r}\right)\sin\theta + A_{11}\left(\frac{1}{r} - \frac{1}{r^3}\right)\sin\theta + A_{13}\left(\frac{1}{r} - \frac{1}{r^3}\right)\sin 3\theta\right] \quad (33)$$

Substituting equation (33) into equation (30) and equating the coefficients of $\frac{1}{r^m}$ and $\cos n\theta$, there result

$$B_{10} = 0$$

$$B_{11} = \frac{M_o}{M_o - 1}\left(A_{11} + A_{13} - 1\right)$$

$$B_{12} = \frac{2M_o}{M_o - 1} A_{13}$$

$$B_{1n} = 0 \quad \text{if} \quad n \geqq 3$$

$$B_{20} = \frac{1}{M_o^2 - 1}\left[\frac{M_o^2}{2}\left(A_{11}^2 + 1 + 5A_{13}^2\right) - \frac{1}{2}\left(B_{11}^2 + B_{12}^2\right)\right]$$

$$B_{21} = \frac{1}{M_o^2 - 1}\left[\frac{M_o^2}{2}\left(-4A_{11} + 4A_{11}A_{13} - 4A_{13}\right) - B_{11}B_{12}\right]$$

$$B_{22} = \frac{1}{M_o^2 - 1}\left[\frac{M_o^2}{2}\left(-8A_{13} + 2A_{11}A_{13}\right) - \frac{1}{2}B_{11}^2\right]$$

$$B_{23} = \frac{1}{M_o^2 - 1}\left[\frac{M_o^2}{2}\left(4A_{13}^2\right) - B_{11}B_{12}\right]$$

$$B_{24} = \frac{1}{M_o^2 - 1} \frac{1}{2} B_{12}^2 H$$

$$B_{2n} = 0 \quad \text{if} \quad n \geq 5$$

$$B_{30} = \frac{1}{M_o^2 - 1} \left[2M_o^2 \left(A_{11} - A_{11}^2 - 3A_{13}^2 \right) - \left(B_{11}B_{21} + B_{12}B_{22} \right) H + \frac{(\gamma + 1)\gamma}{8} B_{11}^2 B_{12} \right]$$

$$B_{31} = \frac{1}{M_o^2 - 1} \left[M_o^2 \left(-A_{11} + A_{11}^2 + 3A_{13} - 6A_{11}A_{13} \right) - \left(2B_{12}B_{20} + B_{11}B_{22} + \right. \right.$$
$$\left. \left. B_{12}B_{21} + B_{12}B_{23} \right) H + \frac{(\gamma + 1)\gamma}{6} \left(\frac{3}{4} B_{11}^3 + \frac{3}{2} B_{11}B_{12}^2 \right) \right]$$

$$B_{32} = \frac{1}{M_o^2 - 1} \left[-\left(B_{11}B_{21} + B_{11}B_{23} + 2B_{12}B_{20} + B_{12}B_{24} \right) H + \frac{(\gamma + 1)\gamma}{4} \left(B_{11}^2 B_{12} + \right. \right.$$
$$\left. \left. \frac{1}{2} B_{12}^3 \right) \right]$$

$$B_{33} = \frac{1}{M_o^2 - 1} \left[-3M_o^2 A_{13}^2 - \left(B_{11}B_{22} + B_{11}B_{24} + B_{12}B_{21} \right) H + \frac{(\gamma + 1)\gamma}{24} \left(B_{11}^3 + \right. \right.$$
$$\left. \left. 3B_{11}B_{12}^2 \right) \right]$$

$$B_{34} = \frac{1}{M_o^2 - 1} \left[-\left(B_{11}B_{23} + B_{12}B_{22} \right) H + \frac{(\gamma + 1)\gamma}{8} B_{11}^2 B_{12} \right]$$

$$B_{35} = \frac{1}{M_o^2 - 1} \left[-\left(B_{11}B_{24} + B_{12}B_{23} \right) H + \frac{(\gamma + 1)\gamma}{8} B_{11}B_{12}^2 \right]$$

$$B_{36} = \frac{1}{M_o^2 - 1} \left[-B_{12}B_{24} H + \frac{(\gamma + 1)\gamma}{24} B_{12}^3 \right]$$

$$B_{3n} = 0 \quad \text{if} \quad n \geq 7$$

$$B_{40} = \frac{1}{M_o{}^2 - 1}\left[\frac{M_o{}^2}{2}\left(5A_{11}{}^2 + 9A_{13}{}^2\right) - H\left(B_{20}{}^2 + \frac{1}{2}B_{21}{}^2 + \frac{1}{2}B_{22}{}^2 + \frac{1}{2}B_{23}{}^2 + \right.\right.$$

$$\frac{1}{2}B_{24}{}^2 + B_{11}B_{31} + B_{12}B_{32}\Big) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{2}B_{11}{}^2B_{20} + \frac{1}{4}B_{11}{}^2B_{22} + \right.$$

$$\frac{1}{2}B_{11}B_{12}B_{21} + \frac{1}{2}B_{11}B_{12}B_{23} + \frac{1}{2}B_{12}{}^2B_{20} + \frac{1}{4}B_{12}{}^2B_{24}\Big) + \frac{(\gamma + 1)\gamma}{24}\left(\frac{3}{8}B_{11}{}^4 + \right.$$

$$\left.\left.\frac{3}{2}B_{11}{}^2B_{12}{}^2 + \frac{3}{8}B_{12}{}^4\right)\right]$$

$$B_{41} = \frac{1}{M_o{}^2 - 1}\left[\frac{M_o{}^2}{2}\left(-4A_{11}{}^2 + 12A_{11}A_{13}\right) - H\left(2B_{20}B_{21} + B_{21}B_{22} + B_{22}B_{23} + \right.\right.$$

$$B_{23}B_{24} + 2B_{11}B_{30} + B_{11}B_{32} + B_{12}B_{31} + B_{12}B_{33}\Big) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{3}{4}B_{12}{}^2B_{21} + \right.$$

$$\frac{1}{4}B_{11}{}^2B_{23} + B_{11}B_{12}B_{20} + B_{11}B_{12}B_{23} + \frac{1}{2}B_{11}B_{12}B_{24} + \frac{1}{2}B_{12}{}^2B_{21} + \right.$$

$$\left.\frac{1}{4}B_{12}{}^2B_{23}\right) + \frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(2B_{11}{}^3B_{12} + \frac{3}{2}B_{11}B_{12}{}^3\right)\Big]$$

$$B_{42} = \frac{1}{M_o{}^2 - 1}\left[-3M_o{}^2A_{11}A_{13} - H\left(\frac{1}{2}B_{21}{}^2 + 2B_{20}B_{22} + B_{21}B_{23} + B_{22}B_{24} + B_{11}B_{31} + \right.\right.$$

$$B_{11}B_{33} + 2B_{12}B_{30} + B_{12}B_{34}\Big) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{2}B_{11}{}^2B_{20} + \frac{1}{2}B_{11}{}^2B_{22} + \frac{1}{4}B_{11}{}^2B_{24} + \right.$$

$$B_{11}B_{12}B_{21} + \frac{1}{2}B_{11}B_{12}B_{23} + \frac{3}{4}B_{12}{}^2B_{22}\Big) + \frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{1}{2}B_{11}{}^4 + \right.$$

$$\left.\left.\frac{9}{4}B_{11}{}^2B_{12}{}^2\right)\right]$$

$$B_{43} = \frac{1}{M_0^2 - 1}\left[-H\left(2B_{20}B_{23} + B_{21}B_{22} + B_{21}B_{24} + 2B_{11}B_{32} + 2B_{11}B_{34} + \right.\right.$$

$$2B_{12}B_{31} + 2B_{12}B_{35}\Big) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{4}B_{11}{}^2B_{21} + \frac{1}{2}B_{11}{}^2B_{23} + \right.$$

$$B_{11}B_{12}B_{20} + \frac{1}{2}B_{11}B_{12}B_{22} + \frac{1}{2}B_{11}B_{12}B_{24} + \frac{1}{4}B_{12}{}^2B_{21} + \frac{1}{2}B_{12}{}^2B_{23}\Big) +$$

$$\left. \frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{3}{2}B_{11}{}^3B_{12} + \frac{3}{2}B_{11}B_{12}{}^3\right)\right]$$

$$B_{44} = \frac{1}{M_0^2 - 1}\left[-H\left(\frac{1}{2}B_{22}{}^2 + 2B_{20}B_{24} + B_{21}B_{23} + B_{11}B_{33} + B_{11}B_{35} + B_{12}B_{33} + \right.\right.$$

$$B_{12}B_{36}\Big) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{4}B_{11}{}^2B_{22} + \frac{1}{2}B_{11}{}^2B_{24} + \frac{1}{2}B_{11}B_{12}B_{21} + \frac{1}{2}B_{11}B_{12}B_{23} + \right.$$

$$\left.\frac{1}{2}B_{12}{}^2B_{20} + \frac{1}{2}B_{12}{}^2B_{24}\right) + \frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{1}{8}B_{11}{}^4 + \frac{3}{2}B_{11}{}^2B_{12}{}^2 + \frac{1}{8}B_{12}{}^4\right)\Big]$$

$$B_{45} = \frac{1}{M_0^2 - 1}\left[-H\left(B_{21}B_{24} + B_{22}B_{23} + B_{11}B_{34} + B_{11}B_{36} + B_{12}B_{33}\right) + \right.$$

$$\frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{4}B_{11}{}^2B_{23} + \frac{1}{2}B_{11}B_{12}B_{22} + \frac{1}{2}B_{11}B_{12}B_{24} + \frac{1}{4}B_{12}{}^2B_{21}\right) +$$

$$\left.\frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{1}{2}B_{11}{}^3B_{12} + \frac{1}{2}B_{11}B_{12}{}^3\right)\right]$$

$$B_{46} = \frac{1}{M_0^2 - 1}\left[-H\left(\frac{1}{2}B_{23}{}^2 + B_{22}B_{24} + B_{11}B_{35} + B_{12}B_{34}\right) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{4}B_{11}{}^2B_{24} + \right.\right.$$

$$\left.\frac{1}{2}B_{11}B_{12}B_{23} + \frac{1}{4}B_{12}{}^2B_{23}\right) + \frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{3}{4}B_{11}{}^2B_{12}{}^2\right)\Big]$$

$$B_{47} = \frac{1}{M_0^2 - 1}\left[-H\left(B_{23}B_{24} + B_{12}B_{35}\right) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{2}B_{11}B_{12}B_{24} + \frac{1}{4}B_{12}{}^2B_{23}\right) + \right.$$

$$\left.\frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{1}{2}B_{11}B_{12}{}^3\right)\right]$$

$$B_{48} = \frac{1}{M_o{}^2 - 1}\left[-H\left(\frac{1}{2}B_{24}{}^2 + B_{12}B_{36}\right) + \frac{(\gamma + 1)\gamma}{2}\left(\frac{1}{4}B_{12}{}^2B_{24}\right) + \right.$$

$$\left. \frac{(\gamma + 1)\gamma(\gamma - 2)}{24}\left(\frac{1}{8}B_{12}{}^4\right)\right] \qquad (34)$$

$$B_{4n} = 0 \quad \text{if} \quad n \geqq 9$$

where $H = \dfrac{M_o{}^2}{2} - \dfrac{\gamma(\gamma + 1) - 2}{2(\gamma - 1)}$.   For given free-stream Mach number and γ, the B_{mn}'s are now expressed in terms of A_{11} and A_{13}.   In the numerical example, 12 parameters are taken in the ρ-series which seems to give satisfactory convergence at high Mach numbers.

The condition of irrotationality is now the only differential equation remaining to be satisfied.   In terms of ψ and polar coordinates, equation (4) becomes

$$\rho_r\psi_r + \frac{\rho_\theta\psi_\theta}{r^2} - \rho(\Delta\psi) = 0 \qquad (35)$$

where

$$\Delta\psi = \psi_{rr} + \frac{\psi_r}{r} + \frac{\psi_{\theta\theta}}{r^2}$$

The Galerkin equation for determining the parameters A_{mn} is:

$$\int_{r=1}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} \left[\rho_r\psi_r + \frac{\rho_\theta\psi_\theta}{r^2} - \rho(\Delta\psi)\right]\frac{\partial\psi}{\partial A_{mn}} \, r \, d\theta \, dr = 0 \qquad (36)$$

With the two parameters A_{11} and A_{13} in ψ, equation (36) becomes:

$$\int_{r=1}^{r=\infty}\int_{\theta=0}^{\theta=2\pi}\left[\rho_r\psi_r + \frac{\rho_\theta\psi_\theta}{r^2} - \rho(\Delta\psi)\right]\left(1 - \frac{1}{r^2}\right)\sin\theta\ d\theta\ dr = 0 \quad (37)$$

$$\int_{r=1}^{r=\infty}\int_{\theta=0}^{\theta=2\pi}\left[\rho_r\psi_r + \frac{\rho_\theta\psi_\theta}{r^2} - \rho(\Delta\psi)\right]\left(1 - \frac{1}{r^2}\right)\sin 3\theta\ d\theta\ dr = 0 \quad (38)$$

Carrying out the integration and taking $\gamma = 1.405$ the following equations are obtained for different values of M_o. The parameters taken in the ρ-series are B_{11}, B_{12}, B_{20}, B_{21}, B_{22}, B_{23}, B_{30}, B_{31}, B_{32}, B_{40}, B_{41}, and B_{42}. With these B_{mn}'s there result at $M_o = 0.3$:

$0.7914166A_{11} + 0.0148886A_{13} = -0.0529632 + 0.0011372A_{11}A_{13} -$

$$0.0290381A_{11}^2 - 0.0353153A_{13}^2 +$$

$$0.0281407A_{11}^2A_{13} - 0.0368037A_{11}A_{13}^2 -$$

$$0.026577A_{11}^3 - 0.0537032A_{13}^3$$

$-0.0416032A_{11} + 2.1113662A_{13} = 0.0565330 - 0.0524788A_{11}A_{13} -$

$$0.0049465A_{11}^2 - 0.0516919A_{13}^2 -$$

$$0.0426662A_{11}^2A_{13} + 0.0614285A_{11}A_{13}^2 +$$

$$0.0182144A_{11}^3 - 0.0776527A_{13}^3$$

At $M_o = 0.4$:

$$0.8939718A_{11} - 0.0293045A_{13} = -0.1386222 - 0.0363370A_{11}A_{13} -$$

$$0.0208205A_{11}^2 - 0.1594151A_{13}^2 +$$

$$0.0557317A_{11}^2A_{13} - 0.1400827A_{11}A_{13}^2 -$$

$$0.0837686A_{11}^3 - 0.1685111A_{13}^3$$

$$-0.1165173A_{11} + 2.2168408A_{13} = 0.1313933 - 0.1462311A_{11}A_{13} -$$

$$0.1092484A_{11}^2 - 0.2046769A_{13}^2 -$$

$$0.053295A_{11}^2A_{13} + 0.3247757A_{11}A_{13}^2 +$$

$$0.0591811A_{11}^3 - 0.1892021A_{13}^3$$

At $M_o = 0.48$:

$$1.0588773A_{11} - 0.0792806A_{13} = -0.3375697 - 0.0768956A_{11}A_{13} -$$

$$0.030879A_{11}^2 - 0.7005271A_{13}^2 +$$

$$0.0580232A_{11}^2A_{13} + 0.362238A_{11}A_{13}^2 -$$

$$0.0786318A_{11}^3 - 0.1904891A_{13}^3$$

$$-0.2295625A_{11} + 2.215112A_{13} = 0.2795736 - 0.7384032A_{11}A_{13} -$$

$$0.2272091A_{11}^2 - 0.9122836A_{13}^2 +$$

$$0.210928A_{11}^2A_{13} + 1.067842A_{11}A_{13}^2 +$$

$$0.1269145A_{11}^3 - 0.1446766A_{13}^3$$

At $M_O = 0.5$:

$$1.0588045A_{11} - 0.0943964A_{13} = -0.4354569 - 0.0744038A_{11}A_{13} -$$

$$0.0687631A_{11}^2 - 1.0282597A_{13}^2 -$$

$$0.0279679A_{11}^2A_{13} + 0.3904425A_{11}A_{13}^2 +$$

$$1.236569A_{11}^3 - 0.5792328A_{13}^3$$

$$-0.2586206A_{11} + 2.1543749A_{13} = 0.3471313 - 0.9668059A_{11}A_{13} -$$

$$0.2896804A_{11}^2 - 1.2930194A_{13}^2 +$$

$$0.4529877A_{11}^2A_{13} + 1.7020295A_{11}A_{13}^2 +$$

$$0.1361617A_{11}^3 - 0.3869221A_{13}^3$$

These equations can be solved by the method of successive approximation as outlined in references 1 and 2. The parameters at various Mach numbers and the computed maximum velocities at these Mach numbers are shown in table I. These values show good agreement with those obtained by other approximate methods. At $M_O = 0.5$ the method of successive approximation fails, and it was found by graphical method that these equations do not have a solution. This agrees with a previous investigation as reported in reference 7 where the velocity potential φ was used. This indicates that the flow is no longer irrotational and shock waves have probably appeared. The limiting free-stream Mach number is between 0.48 and 0.5. This value is probably too high because only two parameters in ψ are used. This fact has been discussed in reference 7.

To get an idea of the convergence of the ψ- and ρ-series, the values of maximum velocities and ρ at $M_O = 0.48$ are computed and listed in tables II, III, and IV. The ρ-series does not appear to be convergent when the parameters B_{mn} are added one by one, but if these parameters are added by the group the series becomes convergent. This can be seen by comparing the values of ρ and the maximum velocities with various numbers of B_{mn}'s as given in tables II and III.

TRANSONIC FLOW PAST A CIRCULAR CYLINDER WITH SHOCK WAVES

When the free-stream Mach number reaches 0.5 there is no solution
to the equations obtained by assuming irrotational flow in the entire
domain. Shock waves probably have occurred. Thus the flow is irrota-
tional only in the region before the shock waves or D_1 and becomes
rotational in the region behind the shock waves D_2. The approximate
solution according to the modified Galerkin method can be carried out
as follows:

(1) The locations and the obliquity of the shock waves are first
assumed from the best evidences available.

(2) When the assumed shock waves are inserted in the flow, the
domains D_1 and D_2 are thus defined. In the region D_1, the flow
is still irrotational. The boundary conditions are that the velocity
at infinity is equal to the undisturbed velocity and on the surface of
the body ψ_1 must be equal to a chosen constant. The values of ψ_1
and ρ_1 may be assumed in the same form as in the case before the shock
waves occur. The parameters A_{mn} and B_{mn} are determined from the
modified Galerkin method. The only difference between the present case
and the previous case is that in this case the integrals are to be
extended only in the region D_1 instead of the entire flow region.

(3) The entropy distribution $S = c_v f(\psi)$ in the flow after the
shock is to be assumed next. From the flow conditions before the occur-
rence of shock waves, a good approximation of $f(\psi)$ may be found by
rough preliminary calculations.

(4) The boundary condition at infinity of the flow after the shock
is that p is equal to the constant pressure p_0. With $f(\psi)$ assumed,
$\rho_0(x,y)$ can be computed from equation (10) and the velocity distribution
from equation (5a).

(5) With the above boundary conditions and the condition that $\psi_1 = \psi_2$
on the assumed shock waves and common streamlines of D_1 and D_2, ψ_2
and ρ_2 are then assumed in the forms of series with undetermined param-
eters. These parameters can again be determined by the modified Galerkin
method.

(6) After these undetermined constants in ψ_1 and ψ_2 are computed,
the values of $\left(1/\rho_1\right)\left(\partial\psi_1/\partial n\right)$ and $\left(1/\rho_2\right)\left(\partial\psi_2/\partial n\right)$ at the shock waves

can be found and $f(\psi)$ is calculated. If $\left(1/\rho_1\right)\left(\partial\psi_1/\partial n\right)$ is not equal to $\left(1/\rho_2\right)\left(\partial\psi_2/\partial n\right)$ and $f(\psi)$ found is not the same as $f(\psi)$ assumed, as is generally the case, adjust the locations and the obliquity of the shock waves. Then based on the new position and shape of the shock waves, assume a new $f(\psi)$ and repeat the computations again until these values check with each other in two consecutive cycles.

A numerical example will now be carried out. With ψ_1 and ρ_1 as assumed in equations (31) and (32) and by taking 2 parameters A_{11} and A_{13} in equation (31) and 12 parameters in equation (32), it was found that there does not exist a solution at $M_0 = 0.5$. In reference 11 it is shown that when the shock waves first occur they are almost normal shocks located near the point of maximum velocity. As a first approximation, assume that at $M_0 = 0.5$ there are two normal shock waves located at $\theta = \pi/2$ and $3\pi/2$ from $r = 1$ to $r = 1.5$. The separating streamlines between the regions D_1 and D_2 are assumed to be the same as the streamlines in the flow at $M_0 = 0.48$ passing through the points $r = 1.5$ and $\theta = \pi/2$, $3\pi/2$.

To determine the parameters A_{11} and A_{13}, integrals (37) and (38) are to be extended in the above assumed region. This can be done by taking first the integration over the entire flow region and then subtracting from the resulting values the values of these integrals evaluated in the region D_2. The region D_2 consists of two areas, namely, a semicircular ring with r from 1 to 1.5 and θ from $\pi/2$ to $-\pi/2$ and a symmetrical tail-shaped area. The dominating parts of the values of the integrals happen to be in the semicircular-ring area where the integration can be carried out analytically. In the tail-shaped area, analytical integration becomes rather laborious and numerical integration has been used. After subtraction, these two equations are

$$2.0587564A_{11} - 0.2021789A_{13} = -0.8773276 - 0.1912757A_{11}A_{13} -$$

$$0.1931520A_{11}{}^2 - 2.010768A_{13}{}^2 +$$

$$0.3490648A_{11}{}^2A_{13} + 0.8948537A_{11}A_{13}{}^2 -$$

$$0.0087952A_{11} + 9.2190418A_{13}{}^3$$

$$-0.3358554A_{11} + 5.2963361A_{13} = 0.7652748 - 1.7476647A_{11}A_{13} -$$

$$0.4624384A_{11}^2 - 2.088238A_{13}^2 +$$

$$0.7361791A_{11}^2A_{13} + 2.9033634A_{11}A_{13}^2 +$$

$$0.3574436A_{11}^3 - 7.1979434A_{13}^3$$

While a solution does not exist before the integrals in D_2 are subtracted, solution of these equations now does exist. Solving, there result:

$$A_{11} = -0.437 \qquad A_{13} = 0.100$$

With these values of A_{11} and A_{13}, the point at $\theta = \pi/2$ with a unit local Mach number is computed to be at $r = 1.6$. This indicates that the shock waves probably would terminate at a value of r between 1.5 and 1.6, and the region D_1 as assumed is a good approximation.

The boundary conditions for ψ_2 and ρ_2 are as follows:

At $r = \infty$, $p = p_0$ and therefore

$$\rho_2^\gamma = \frac{p_0}{c_3} e^{-f(\psi_2)} \tag{39}$$

At $r = 1$, since ψ_1 is taken as zero,

$$\psi_2 = 0 \tag{40}$$

To insure the vanishing of normal velocity,

$$\psi_{2\theta} = 0 \tag{41}$$

On the shock waves and the common boundary,

$$\psi_2 = \psi_1 \tag{42}$$

Conditions (39), (40), and (41) are satisfied if ψ_2 and ρ_2 are assumed in the following forms:

$$\psi_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A'_{mn}\left(\frac{1}{r^m} - \frac{1}{r^{m+2}}\right) \sin n\theta + \left(r - \frac{1}{r}\right) \sin \theta \sum_{m=1}^{\infty} C_m r^m \sin^m\theta \quad (43)$$

and

$$\rho_2 = \left[\frac{p_o}{c_3} e^{-f\left(\psi_2\right)}\right]_{r=\infty}^{1/\gamma} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B'_{mn} \frac{\cos 2n\theta}{r^{2m}} \quad (44)$$

Note that ψ_2 becomes a function of $y = r \sin \theta$ at $r = \infty$. The constants C_m are to be determined by the collation method in terms of A'_{mn} so that $\psi_2 = \psi_1$ at a number of points on the shock waves and the common boundary. With $f\left(\psi_2\right)$ assumed, B'_{mn} may be solved in terms of A'_{mn} from equation (24) and the parameters A'_{mn} may be calculated from equation (26) by using Galerkin's method. A numerical example has been tried. It was found, however, that unless some modern high-speed computing machine is used the computation becomes very lengthy.

New York University
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TABLE I

PARAMETERS AND COMPUTED MAXIMUM VELOCITIES

AT VARIOUS MACH NUMBERS

M_o	0.3	0.4	0.47	0.48
A_{11}	-0.0677	-0.1537	-0.296	-0.316
A_{13}	0.0255	0.0404	0.088	0.087
$\dfrac{\rho}{\rho_o}$ at $\begin{array}{c} r = 1 \\ \theta = 90^o \end{array}$	0.8254	0.6972	0.3705	0.2680
$\dfrac{q}{U}$ at $\begin{array}{c} r = 1 \\ \theta = 90^o \end{array}$	2.1972	2.2859	3.4870	4.5525

TABLE II

CONVERGENCE OF ρ-SERIES AT $M_o = 0.48$

Parameters taken in ρ series	B_{11}	$B_{11}B_{12}$	$B_{11}B_{12}$ B_{20}	$B_{11}B_{12}$ $B_{20}B_{21}$	$B_{11}B_{12}$ $B_{20}B_{21}$ $B_{22}B_{23}$	$B_{11}B_{12}$ $B_{20}B_{21}$ $B_{22}B_{23}$ $B_{30}B_{31}$ B_{32}	$B_{11}B_{12}$ $B_{20}B_{21}$ $B_{22}B_{23}$ $B_{30}B_{31}$ $B_{32}B_{40}$ $B_{41}B_{42}$
A_{11}	-0.0988	-0.0836	-0.490	-0.390	-0.3950	-0.3547	-0.316
A_{13}	0.0777	0.0784	0.120	0.091	0.0960	0.1110	0.074
$\dfrac{\rho}{\rho_o}$ at $r = 1$ $\theta = 90°$	0.6943	0.6522	0.1416	0.2931	0.2686	0.2788	0.2680
$\dfrac{q}{U}$ at $r = 1$ $\theta = 90°$	2.3765	2.5698	5.5104	3.5416	3.7902	4.5503	4.5525

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TABLE III

CONVERGENCE OF ρ-SERIES AT $M_O = 0.48$

Number of parameters taken in ρ-series (1)	1	2	3	4	5	6	7	8	9	10	11	12
$\frac{\rho}{\rho_0}$ at $r = 1$ $\theta = 90°$	0.6943	0.6522	0.1416	0.2931	0.2686	0.2558	0.5187	0.3853	0.2788	0.0378	0.4520	0.2680
$\frac{q}{U}$ at $r = 1$ $\theta = 90°$	2.3765	2.5698	5.5104	3.5416	3.7902	3.9797	2.4557	3.2925	4.5503	25.0000	2.6966	4.5525

[1]Parameters in ρ-series are taken according to the following order: B_{11}, B_{12}, B_{20}, B_{21}, B_{22}, B_{23}, B_{30}, B_{31}, B_{32}, B_{40}, B_{41}, B_{42}.

TABLE IV

CONVERGENCE OF ψ SERIES WITH

12 PARAMETERS IN ρ-SERIES AT $M_o = 0.48$

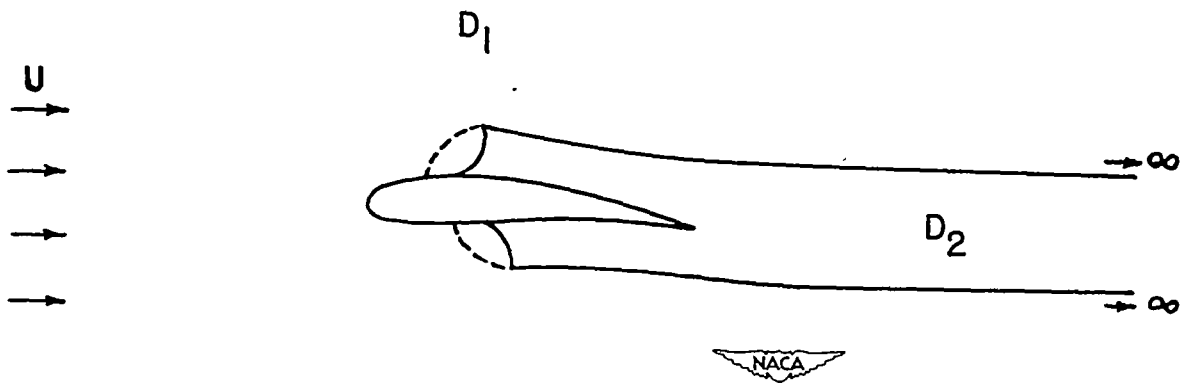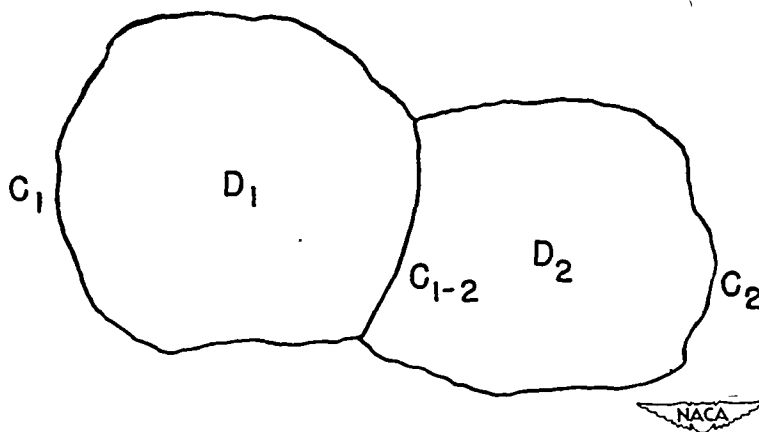| Parameters taken in ψ-series	A_{11}	$A_{11}A_{13}$
$\dfrac{\rho}{\rho_o}$ at $\begin{array}{l} r = 1 \\ \theta = 90^o \end{array}$	0.1007	0.2680
$\dfrac{q}{U}$ at $\begin{array}{l} r = 1 \\ \theta = 90^o \end{array}$	13.5194	4.5525

Figure 1.- Flow past an airfoil.

Figure 2.- Simplified sketch of flow past an airfoil.